

Resonance and rapid decay of exponential sums of Fourier coefficients of a Maass form for $GL_m(\mathbb{Z})$

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Abstract Let f be a full-level cusp form for $GL_m(\mathbb{Z})$ with Fourier coefficients $A_f(n_1, \dots, n_{m-1})$. In this paper an asymptotic expansion of Voronoi's summation formula for $A_f(n_1, \dots, n_{m-1})$ is established. As applications of this formula, a smoothly weighted average of $A_f(n, 1, \dots, 1)$ against $e(\alpha|n|^\beta)$ is proved to be rapidly decayed when $0 < \beta < 1/m$. When $\beta = 1/m$ and α equals or approaches $\pm mq^{1/m}$ for a positive integer q , this smooth average has a main term of the size of $|A_f(1, \dots, 1, q) + A_f(1, \dots, 1, -q)|X^{1/(2m)+1/2}$, which is a manifestation of resonance of oscillation exhibited by the Fourier coefficients $A_f(n, 1, \dots, 1)$. Similar estimate is also proved for a sharp-cut sum.

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1. INTRODUCTION

Voronoi's summation formula is an important technique in analytic number theory. Ivić [1] generated the original Voronoi formula to non-cuspidal forms as given by multiple divisor functions. For cuspidal representations of $GL_2(\mathbb{Z})$, a Voronoi-type summation formula was proved by Sarnak [2] for holomorphic cusp forms and by Kowalski-Michel-Vanderkam [3] for Maass forms. Voronoi's summation formula for Maass forms for $SL_3(\mathbb{Z})$ was proved by Miller-Schmid [4] and Goldfeld-Li [5]. For $m \geq 4$, the Voronoi summation formula was proved by Miller-Schmid [6]. To state the formula, let f be a full-level cusp form for $GL_m(\mathbb{Z})$ with Langlands' parameters $\mu_f(j)$, $j = 1, \dots, m$, and Fourier coefficients $A_f(c_{m-2}, \dots, c_1, n)$. Let $\psi \in C_c^\infty(\mathbb{R}^+)$, q a positive integer, and h an integer coprime with q . Let $h\bar{h} \equiv 1 \pmod{q}$. Then Voronoi summation formula as proved by Miller-Schmid [6] is the following

$$\begin{aligned}
& \sum_{n \neq 0} A_f(c_{m-2}, c_{m-3}, \dots, c_1, n) e\left(-\frac{nh}{q}\right) \psi(|n|) \\
&= q \sum_{d_1 | c_1 q} \sum_{d_2 | \frac{c_1 c_2 q}{d_1}} \cdots \sum_{d_{m-2} | \frac{c_1 \cdots c_{m-2} q}{d_1 d_2 \cdots d_{m-3}}} \sum_{n \neq 0} \frac{A_f(n, d_{m-2}, \dots, d_1)}{d_1 \cdots d_{m-2} |n|} \\
&\times S(n, \bar{h}; q, c, d) \Psi\left(\frac{|n|}{q^m} \prod_{i=1}^{m-2} \frac{d_i^{m-i}}{c_i^{m-i-1}}\right), \tag{1.1}
\end{aligned}$$

where $c = (c_1, \dots, c_{m-2})$, $d = (d_1, \dots, d_{m-2})$, and

$$\begin{aligned}
S(n, \bar{h}; q, c, d) &= \sum_{x_1 \pmod{\frac{c_1 q}{d_1}}}^* e\left(\frac{d_1 x_1 n}{q}\right) \sum_{x_2 \pmod{\frac{c_1 c_2 q}{d_1 d_2}}}^* e\left(\frac{d_2 x_2 \bar{x}_1}{\frac{c_1 q}{d_1}}\right) \cdots \\
&\times \sum_{x_{m-2} \pmod{\frac{c_1 \cdots c_{m-2} q}{d_1 \cdots d_{m-2}}}}^* e\left(\frac{d_{m-2} x_{m-2} \bar{x}_{m-3}}{\frac{c_1 \cdots c_{m-3} q}{d_1 \cdots d_{m-3}}} + \frac{\bar{h} \bar{x}_{m-2}}{\frac{c_1 \cdots c_{m-2} q}{d_1 \cdots d_{m-2}}}\right). \tag{1.2}
\end{aligned}$$

The $*$ in $\sum_{t \pmod{r}}^*$ indicates that $(t, r) = 1$. Here Ψ is an integral transform of ψ given by

$$\Psi(x) = \frac{1}{2\pi i} \int_{\Re s = -\sigma} \tilde{\psi}(s) x^s \frac{\tilde{F}(1-s)}{F(s)} ds, \tag{1.3}$$

where

$$\tilde{\psi}(s) = \int_0^\infty \psi(x) x^s \frac{dx}{x}$$

and

$$F(s) = \pi^{-ms/2} \prod_{i=1}^m \Gamma\left(\frac{s - \mu_f(j)}{2}\right), \tag{1.4}$$

$$\tilde{F}(s) = \pi^{-ms/2} \prod_{i=1}^m \Gamma\left(\frac{s - \bar{\mu}_f(j)}{2}\right) \tag{1.5}$$

with $\{\bar{\mu}_f(j)\}_{j=1, \dots, m} = \{\mu_{\tilde{f}}(j)\}_{j=1, \dots, m}$ being the Langlands' parameters for the dual form \tilde{f} of f .

A special case of (1.1) for even Maass forms for $SL_m(\mathbb{Z})$ was proved by Goldfeld-Li [5, 7, 8]:

$$\begin{aligned}
& \sum_{n \neq 0} A_f(1, 1, \dots, 1, n) e\left(\frac{nh}{q}\right) \psi(|n|) \\
&= q \sum_{d_1|q} \sum_{d_2|\frac{q}{d_1}} \cdots \sum_{d_{m-2}|\frac{q}{d_1 d_2 \cdots d_{m-3}}} \sum_{n \neq 0} \frac{A_f(n, d_{m-2}, \dots, d_1)}{d_1 \cdots d_{m-2} |n|} \\
&\times KL(\bar{h}, n; d, q) \Psi\left(\frac{|n|}{q^m} \prod_{i=1}^{m-2} d_i^{m-i}\right), \tag{1.6}
\end{aligned}$$

where $KL(\bar{h}, n; d, q)$ is the Kloosterman sum

$$\begin{aligned}
KL(\bar{h}, n; d, q) &= \sum_{t_1 \pmod{\frac{q}{d_1}}}^* e\left(\frac{\bar{h} t_1}{\frac{q}{d_1}}\right) \sum_{t_2 \pmod{\frac{q}{d_1 d_2}}}^* e\left(\frac{\bar{t}_1 t_2}{\frac{q}{d_1 d_2}}\right) \cdots \\
&\times \sum_{t_{m-2} \pmod{\frac{q}{d_1 \cdots d_{m-2}}}}^* e\left(\frac{\bar{t}_{m-3} t_{m-2}}{\frac{q}{d_1 \cdots d_{m-2}}}\right) e\left(\frac{n \bar{t}_{m-2}}{\frac{q}{d_1 \cdots d_{m-2}}}\right). \tag{1.7}
\end{aligned}$$

Note that (1.7) can be rewritten as (1.2). When $q = 1$, the formulas (1.1) and (1.6) become

$$\sum_{n \neq 0} A_f(1, \dots, 1, n) \psi(|n|) = \sum_{n \neq 0} \frac{A_f(n, 1, \dots, 1)}{|n|} \Psi(|n|). \tag{1.8}$$

Replacing f by its dual form \tilde{f} and noting that $A_f(n, 1, \dots, 1) = A_{\tilde{f}}(1, \dots, 1, n)$, we have

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) \psi(|n|) = \sum_{n \neq 0} \frac{A_f(1, \dots, 1, n)}{|n|} \tilde{\Psi}(|n|), \tag{1.9}$$

where $\tilde{\Psi}(x)$ is defined as $\Psi(x)$ in (1.3) by replacing $\mu_f(j)$ by $\mu_{\tilde{f}}(j)$ in (1.4) and (1.5).

In applications, asymptotic behavior of $\Psi(x)$ is often required. An asymptotic expansion for Voronoi's summation formula were firstly obtained by Ivić [1] for multiple divisor functions. Similar asymptotic formulas were proved and used in subconvexity bounds for GL_2 L -functions by Sarnak [2] for holomorphic cusp forms and by Liu-Ye [9] and Lau-Liu-Ye [10] for Maass forms. For Maass forms for $SL_3(\mathbb{Z})$, an asymptotic Voronoi formula was proved by Li [11] and Ren-Ye [12] and applied to subconvexity problems for L -functions attached to a self-dual Maass form for $SL_3(\mathbb{Z})$ by Li [13].

In this paper we prove the following asymptotic expansion.

Theorem 1.1. *Let f be a full-level cusp form for $GL_m(\mathbb{Z})$. Let $m \geq 3$ be an integer. Let $\Psi(x)$ be as defined in (1.3) with $\psi(y) = \phi(y/X)$, where $\phi(x) \ll 1$ is a fixed smooth function of compact support on $[a, b]$ with $b > a > 0$. Then for any $x > 0$, $xX \gg 1$ and $r > m/2$, we*

have

$$\begin{aligned}
\Psi(x) &= x \sum_{k=0}^r c_k \int_0^\infty (xy)^{1/(2m)-1/2-k/m} \psi(y) \\
&\times \left\{ i^{k+(m-1)/2} e(m(xy)^{1/m}) + (-i)^{k+(m-1)/2} e(-m(xy)^{1/m}) \right\} dy \\
&+ O((xX)^{-r/m+1/2+\varepsilon}),
\end{aligned} \tag{1.10}$$

where c_k , $k = 0, \dots, r$, are constants depending on m and $\{\mu_f(j)\}_{j=1, \dots, m}$ with $c_0 = -1/\sqrt{m}$, and the implied constant depends at most on f , ϕ , r , a , b and ε .

We point out that if we replace f by \tilde{f} in (1.3), the formula (1.10) for $\tilde{\Psi}(x)$ has the same leading term, i.e., $\tilde{c}_0 = c_0 = -1/\sqrt{m}$.

As applications of Theorem 1.1, we consider sums of the Fourier coefficients $A_f(n, 1, \dots, 1)$ twisted with the exponential function $e(\pm\alpha|n|^\beta)$. Our results are the following.

Theorem 1.2. *Let f be a full-level cusp form for $GL_m(\mathbb{Z})$ and $\phi(x)$ a C^∞ function on $(0, \infty)$ of compact support $[1, 2]$ with $\phi^{(j)}(x) \ll 1$ for $j \geq 1$. Let $X > 1$ and $\alpha, \beta \geq 0$.*

(i) *Suppose*

$$2 \max\{1, 2^{\beta-1/m}\} (\alpha\beta)^m \leq X^{1-\beta m}. \tag{1.11}$$

Then the estimate

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\pm\alpha|n|^\beta) \phi\left(\frac{|n|}{X}\right) \ll X^{-M} \tag{1.12}$$

holds for any $M > 0$, where the implied constant in (1.12) depends on m , f , β and M .

(ii) *Suppose*

$$2 \max\{1, 2^{\beta-1/m}\} (\alpha\beta)^m > X^{1-\beta m}. \tag{1.13}$$

Then for $\beta \neq 1/m$, we have

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\pm\alpha|n|^\beta) \phi\left(\frac{|n|}{X}\right) \ll_{m, f, \beta} (\alpha X^\beta)^{m/2}.$$

For $\beta = 1/m$, we have

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\pm\alpha|n|^\beta) \phi\left(\frac{|n|}{X}\right) \ll_{m, f} (\alpha X^\beta)^{(m+1)/2}. \tag{1.14}$$

Moreover, for $X > \alpha^{m(m-1)/(1-m\varepsilon)}$ with $0 < \varepsilon < 1/m$,

$$\begin{aligned}
& \sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\pm \alpha |n|^{1/m}) \phi\left(\frac{|n|}{X}\right) \\
&= m \{A_f(1, 1, \dots, 1, n_\alpha) + A_f(1, 1, \dots, 1, -n_\alpha)\} \sum_{k=0}^r \rho_\pm(k, m, \alpha, X) X^{1/(2m)+1/2-k/m} \\
&\quad + O_{m,r,\varepsilon}(X^{-r/m+1/2+\varepsilon}), \tag{1.15}
\end{aligned}$$

where

$$\begin{aligned}
\rho_\pm(k, m, \alpha, X) &= (\mp i)^{k+(m-1)/2} \tilde{c}_k n_\alpha^{1/(2m)-1/2-k/m} \\
&\quad \times \int_0^\infty t^{m/2-k-1/2} \phi(t^m) e((\alpha - mn_\alpha^{1/m})X^{1/m}t) dt
\end{aligned}$$

with n_α being the integer satisfying $(\alpha/m)^m - n_\alpha \in (-1/2, 1/2]$, \tilde{c}_k being constants depending on m and f .

We remark that the condition (1.11) holds for any fixed α and $\beta \in (0, 1/m)$ when X is large enough in terms of α and β , and hence the rapid decay in (1.12) holds. For $\beta = 1/m$ this is the case when $0 \leq \alpha \leq m2^{-1/m}$. For $\beta > 1/m$, (1.12) holds for $0 \leq \alpha \leq 2mX^{1/m-\beta}$. When $\alpha = 0$ this result was firstly obtained by Booker [14] via a different approach.

Note that the main term in (1.15) is negligible when $|(\alpha/m)^m - n_\alpha| > X^{\varepsilon-1/m}$ since the integral in $\rho_\pm(k, m, \alpha, X)$ is arbitrarily small by repeated partial integrating by parts. Consequently, (1.15) manifests resonance of Fourier coefficients of f against $e(\pm \alpha n^{1/m})$ when α approaches $mq^{1/m}$. In particular, when $\alpha = mq^{1/m}$ we obtain the following.

Corollary 1.1. *Let q be a positive integer and $0 < \varepsilon < 1/m$. Then for $X > (m^m q)^{(m-1)/(1-m\varepsilon)}$ we have*

$$\begin{aligned}
& \sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\pm m(q|n|)^{1/m}) \phi\left(\frac{|n|}{X}\right) \\
&= \{A_f(1, \dots, 1, q) + A_f(1, \dots, 1, -q)\} \sum_{k=0}^r \omega_\pm(k, m, q) X^{1/(2m)+1/2-k/m} \\
&\quad + O_{m,r,\varepsilon}(X^{-r/m+1/2+\varepsilon}),
\end{aligned}$$

where

$$\omega_\pm(k, m, q) = (\mp i)^{k+(m-1)/2} \tilde{c}_k q^{1/(2m)-1/2-k/m} \int_0^\infty x^{1/(2m)-1/2-k/m} \phi(x) dx.$$

The asymptotic behavior as described in Corollary 1.1 was proved for cusp forms for $SL_2(\mathbb{Z})$ in Iwaniec-Luo-Sarnak [15] and Ren-Ye [16], while for Maass forms for $SL_3(\mathbb{Z})$ it was proved in Ren-Ye [17]. For similar results concerning coefficients of L -functions for general Selberg class, see Kaczorowski and Perelli [18] [19].

We will also prove a sharp-cut version of Theorem 1.2.

Theorem 1.3. *Let f be a full-level cusp form for $GL_m(\mathbb{Z})$. Let $X > 1$ and $\alpha, \beta \geq 0$.*

(i) *Assume that (1.11) holds. Then we have*

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\pm \alpha |n|^\beta) \ll_{f, \beta, m} X^{1-1/m}. \quad (1.16)$$

(ii) *Assume that (1.13) holds. Then for $\beta \neq 1/m$ we have*

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\pm \alpha |n|^\beta) \ll_{f, \beta, m} (\alpha X^\beta)^{m/2} + X^{1-1/m}. \quad (1.17)$$

For $\beta = 1/m$ we have

$$\begin{aligned} & \sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\pm \alpha |n|^{1/m}) \\ & \ll \alpha^{1/2} X^{1/(2m)+1/2} + \alpha^{m-1/2} X^{1/2-1/(2m)} + X^{1-1/m}. \end{aligned} \quad (1.18)$$

Theorem 1.4. *Assume the following bound toward the Ramanujan conjecture*

$$A_f(1, \dots, 1, n) \ll n^\theta, \quad \text{for some } 0 < \theta < \frac{2}{m-1}.$$

(i) *Suppose that the parameters α, β, X satisfy (1.11). Then we have*

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \ll_m X^{(m-1)(1+\theta)/(m+1)}. \quad (1.19)$$

(ii) *Suppose that the parameters α, β, X satisfy (1.13). Then we have, for $\beta \neq 1/m$,*

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \ll (\alpha X^\beta)^{m/2} + X^{(m-1)(1+\theta)/(m+1)}, \quad (1.20)$$

and for $\beta = 1/m$,

$$\begin{aligned} & \sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\pm \alpha |n|^{1/m}) \\ & = -\sqrt{m} X^{1/(2m)+1/2} (-i)^{(m-1)/2} I(m, \alpha, X) \frac{A_f(1, \dots, 1, n_\alpha) + A_f(1, \dots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} \\ & \quad + O(\alpha^{m-1/2} X^{1/2-1/(2m)}) + O(X^{(m-1)(1+\theta)/(m+1)}), \end{aligned} \quad (1.21)$$

where n_α is the integer satisfying $(\alpha/m)^m - n_\alpha \in (-1/2, 1/2]$ and

$$I(m, \alpha, X) = \int_1^{2^{1/m}} t^{m/2-1/2} e((\alpha - m n_\alpha^{1/m}) X^{1/m} t) dt.$$

Note that for fixed α and large X , (1.21) is an asymptotic formula for $\theta < 1/3$ when $m = 3$ and for $\theta < 1/24$ when $m = 4$. Recall that the best known results are $\theta = 5/14$ for $m = 3$

and $\theta = 9/22$ for $m = 4$, by Kim and Sarnak [20][21] and Sarnak [22] (21). When $m \geq 5$, (1.21) implies the following bound

$$\begin{aligned} & \sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^{1/m}) \\ & \ll \alpha^{1/2 + (\theta - 1/2)m} X^{1/(2m) + 1/2} + \alpha^{m-1/2} X^{1/2 - 1/(2m)} + X^{(m-1)(1+\theta)/(m+1)}. \end{aligned}$$

We point out that when $\theta < 1/m$, the bounds in (1.19)-(1.21) improves the bounds in (1.16)-(1.18), respectively.

We will prove Theorem 1.1 in Sections 2-4, and prove Theorems 1.2-1.4 in Sections 5-6.

2. STIRLING'S ASYMPTOTIC

Changing s to $1 - 2s$ and noting that $\{\mu_f(j)\} = \{\bar{\mu}_f(j)\}$ for cusp form f , we get

$$\Psi(x) = i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma} (\pi^m x)^{-2s+1} \tilde{\psi}(-2s+1) G(s) ds, \quad (2.1)$$

where

$$G(s) = \prod_{j=1}^m \frac{\Gamma(s - \frac{\bar{\mu}_f(j)}{2})}{\Gamma(-s + \frac{1 - \mu_f(j)}{2})}. \quad (2.2)$$

Note that the Γ -functions in the numerator on the right side of (2.2) are analytic and nonzero for

$$\sigma > \frac{1}{2} \max \{ \operatorname{Re} \bar{\mu}_f(1), \dots, \operatorname{Re} \bar{\mu}_f(m) \}.$$

A bound due to Luo, Rudnick and Sarnak [23] asserts that

$$|\operatorname{Re} \mu_f(j)| \leq \frac{1}{2} - \frac{1}{m^2 + 1}, \quad j = 1, \dots, m. \quad (2.3)$$

Thus we are allowed to take any $\sigma > 1/4 - 1/(2(m^2 + 1))$ in (2.1) where the convergence of the integral is guaranteed by the rapidly decay of $\tilde{\psi}(-2s+1)$ with respect to t . Let us consider $s = \sigma + it$ with

$$\sigma > \frac{1}{4} - \frac{1}{2(m^2 + 1)}. \quad (2.4)$$

Let $r > m/2$ and define

$$\sigma(r) = \frac{1}{4} + \frac{r}{2m} - \varepsilon, \quad (2.5)$$

where $\varepsilon > 0$ is a small number. We will show that in the vertical strip

$$L_r : |\operatorname{Re} s - \sigma(r)| \leq \frac{\varepsilon}{2},$$

there holds

$$G(s) = m^{-2ms+m/2} \cdot \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \cdot \left(1 + \sum_{j=1}^r \frac{h_j}{s^j} + E_r(s)\right), \quad (2.6)$$

where h_j 's are constants depending on f , $E_r(s)$ is analytic in L_r and satisfies $E_r(s) = O(|s|^{-r-1})$ as $|s| \rightarrow \infty$.

By Stirling's formula [12] [24], for $|\operatorname{Im} s| \geq 2|\beta|$ and $|\operatorname{Im} s| \gg |s|$ one has

$$\log \Gamma(s + \beta) = \left(s + \beta - \frac{1}{2}\right) \log s - s + \log \sqrt{2\pi} + \sum_{j=1}^r \frac{d_j}{s^j} + O(|s|^{-r-1}), \quad (2.7)$$

where d_j are constants depending on β , and the implied constant depends on β and r . This in combination with the fact that $\mu_f(1) + \cdots + \mu_f(m) = 0$ show that, for s satisfying

$$|\operatorname{Im} s| \gg |s| \quad \text{and} \quad |\operatorname{Im} s| \geq t_0 = 2 + \max_{1 \leq j \leq m} \{|\mu_f(j)|\}, \quad (2.8)$$

there holds

$$\begin{aligned} \log G(s) &= \left(ms - \frac{m}{2}\right) \log s - 2ms + ms \log(-s) \\ &\quad + \sum_{j=1}^r \frac{f_j}{s^j} + O(|s|^{-r-1}), \end{aligned} \quad (2.9)$$

where f_j are constants depending on m and $\mu_f(j)$, $j = 1, \dots, m$. Similarly, by (2.7), for $|\operatorname{Im} s| \geq 1$, we have

$$\begin{aligned} \log \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} &= \left(ms - \frac{m}{2}\right) \log ms - 2ms + ms \log(-ms) \\ &\quad + \sum_{j=1}^r \frac{g_j}{s^j} + O(|s|^{-r-1}). \end{aligned} \quad (2.10)$$

Comparing (2.9) with (2.10) we get, for s satisfying (2.8), that

$$G(s) = m^{-2ms+m/2} \cdot \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \cdot \left(1 + \sum_{j=1}^r \frac{h_j}{s^j} + O(|s|^{-r-1})\right). \quad (2.11)$$

On the other hand, one can write

$$G(s) = m^{-2ms+m/2} \cdot \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \cdot A(s), \quad (2.12)$$

where $A(s)$ is analytic and non-zero for $s = \sigma + it$ satisfying (2.4) and

$$\begin{aligned} -ms + \frac{m-1}{2} &\notin \mathbb{N}, \quad ms - \frac{1}{2} \notin \mathbb{N}, \\ -\left(s - \frac{\bar{\mu}_f(j)}{2}\right) &\notin \mathbb{N}, \quad s - \frac{1 - \mu_f(j)}{2} \notin \mathbb{N}, \quad 1 \leq j \leq m. \end{aligned} \quad (2.13)$$

Here we have to avoid poles and zeros of the quotients of Γ -functions on both sides of (2.12). Let $\sigma(r)$ be defined by (2.5). We show that when $\varepsilon > 0$ is small enough (2.13) is satisfied for $s = \sigma + it$ with $|\sigma - \sigma(r)| \leq \varepsilon/2$. Actually, let $\|a\|$ denote the distance from a to the nearest integer. One can easily check that $-ms + (m-1)/2 \notin \mathbb{N}$, $ms - 1/2 \notin \mathbb{N}$ and $-(s - \bar{\mu}_f(j)/2) \notin \mathbb{N}$ when $\varepsilon < 1/(8m)$ and $|\sigma - \sigma(r)| \leq \varepsilon/2$. Moreover, one has

$$\left\| \sigma(r) - \frac{1 - \operatorname{Re} \mu_f(j)}{2} \right\| = \left\| \frac{r}{2m} - \frac{1}{4} + \frac{\operatorname{Re} \mu_f(j)}{2} - \varepsilon \right\|. \quad (2.14)$$

If $m \nmid r$, the right expression in (2.14) is $\geq 1/2(m^2 + 1) - \varepsilon > \varepsilon/2$ when $\varepsilon < 1/4(m^2 + 1)$. For $m \nmid r$, write

$$\delta_j = \left\| \frac{r}{2m} - \frac{1}{4} + \frac{\operatorname{Re} \mu_f(j)}{2} \right\|.$$

Note that when $\delta_j = 0$, the right expression in (2.14) is $\geq \varepsilon$ for any $0 < \varepsilon < 1/2$. If $\delta_j \neq 0$, the expression is $> \varepsilon$ when $\varepsilon < \delta_j/2$. Thus one can choose

$$0 < \varepsilon < \min_{\substack{1 \leq j \leq m \\ \delta_j \neq 0}} \left\{ \frac{1}{4(m^2 + 1)}, \frac{\delta_j}{2} \right\}$$

so that (2.13) is satisfied for $|\sigma - \sigma(r)| \leq \varepsilon/2$.

Now we consider the region

$$D = \left\{ s = \sigma + it \mid |\sigma - \sigma(r)| \leq \frac{\varepsilon}{2}, |t| \leq |\sigma(r)| + 2t_0 \right\}.$$

In this region one can express $A(s)$ as

$$1 + \sum_{j=1}^r \frac{h_j}{s^j} + E_r(s),$$

where $E_r(s)$ is analytic in D . Back to (2.12) and (2.11), we see that (2.6) holds in the vertical strip $L_r = \{\sigma + it : |\sigma - \sigma(r)| \leq \varepsilon/2\}$.

Write $u = \pi^m x$ and define

$$\begin{aligned} \mathcal{H}_j &= i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma(r)} \frac{\Gamma(ms - \frac{m-1}{2})}{s^j \Gamma(-ms + \frac{1}{2})} \\ &\times m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \mathcal{E}_r &= i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma(r)} \frac{m^{-2ms+m/2} \Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \\ &\times E_r(s) u^{-2s+1} \tilde{\psi}(-2s+1) ds. \end{aligned} \quad (2.16)$$

Then we get

$$\Psi(x) = \mathcal{H}_0 + \sum_{j=1}^r h_j \mathcal{H}_j + \mathcal{E}_r.$$

We remark that the integral for \mathcal{E}_r is only valid in the vertical strip L_r , while the integral path for \mathcal{H}_j can be moved freely in the half plane $\operatorname{Re} s > 1/2 - 1/(2m)$. In the following we will replace $\sigma(r)$ by $\sigma_1(r)$ in (2.15) where

$$\sigma_1(r) = \frac{1}{2} - \frac{1}{2m} + \frac{r}{m} + \varepsilon. \quad (2.17)$$

The estimate of \mathcal{E}_r is immediate. By Stirling's formula, for $s = \sigma + it$ we have

$$\frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \ll |s|^{2m\sigma - m/2}, \quad \text{as } |s| \rightarrow \infty.$$

When $r > m/2$, $\sigma = \sigma(r)$ as defined in (2.5) one has

$$\frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} E_r(s) \ll |s|^{2m\sigma(r) - m/2 - r - 1} \ll |s|^{-1 - \varepsilon}.$$

Note that

$$\tilde{\psi}(s) = \int_{aX}^{bX} \psi(y) y^s \frac{dy}{y} \ll X^\sigma \int_a^b |\phi(x)| x^{\sigma-1} dx \ll_\phi X^\sigma. \quad (2.18)$$

Hence

$$\begin{aligned} \mathcal{E}_r &\ll u^{-2\sigma(r)+1} X^{-2\sigma(r)+1} \int_{\operatorname{Re} s = \sigma(r)} |s|^{-1-\varepsilon} |ds| \\ &\ll (uX)^{-2\sigma(r)+1} \ll (xX)^{-r/m+1/2+2\varepsilon}. \end{aligned}$$

This shows

$$\Psi(x) = \mathcal{H}_0 + \sum_{j=1}^r h_j \mathcal{H}_j + O((xX)^{-r/m+1/2+\varepsilon}). \quad (2.19)$$

To prove Theorem 1.1 it remains to compute \mathcal{H}_j for $j = 0, 1, \dots, r$, this will be carried out in the following two sections according to m is odd or even.

3. PROOF OF THEOREM 1.1 WHEN m IS ODD

By definition we have

$$\mathcal{H}_0 = i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds.$$

Move the integral path to $\operatorname{Re} s = -\infty$ and note that the only poles of the integrand are from $\Gamma(ms - (m-1)/2)$ and hence are simple at $s = -n/m + (m-1)/(2m)$ for $n = 0, 1, \dots$ with

the residue $(-1)^n/(n!m)$. Thus

$$\begin{aligned}\mathcal{H}_0 &= -2\pi^{-m/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-\frac{m}{2})} m^{2n-m/2} u^{(2n+1)/m} \tilde{\psi}\left(\frac{2n+1}{m}\right) \\ &= -2\pi^{-m/2} \int_0^\infty \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-\frac{m}{2})} m^{2n-m/2} (uy)^{(2n+1)/m} \psi(y) \frac{dy}{y}.\end{aligned}$$

Recall the power series definition of the Bessel function

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\nu)} \left(\frac{z}{2}\right)^{2n+\nu}, \quad (3.1)$$

we get

$$\begin{aligned}\mathcal{H}_0 &= -2\pi^{-m/2} u \int_0^\infty \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-\frac{m}{2})} \\ &\quad \times (uy)^{1/m-1/2} (m(uy)^{1/m})^{2n-m/2} \psi(y) dy \\ &= -2\pi^{-m/2} u \int_0^\infty (uy)^{1/m-1/2} J_{-m/2}(2m(uy)^{1/m}) \psi(y) dy.\end{aligned} \quad (3.2)$$

Note that if $\nu = -q - 1/2$ and $q \geq 1$, the half integral-order Bessel function is elementary. Applying (8.462.2) in [25], we have

$$\begin{aligned}J_{-q-1/2}(z) &= \frac{1}{\sqrt{2\pi z}} \sum_{j=0}^q \frac{(q+j)!}{j!(q-j)!(2z)^j} \\ &\quad \times \left\{ i^{j+q} e\left(\frac{z}{2\pi}\right) + (-i)^{j+q} e\left(-\frac{z}{2\pi}\right) \right\}.\end{aligned} \quad (3.3)$$

When $m = 2\ell + 1$, we apply (3.3) with $z = 2m(uy)^{1/m} = 2\pi m(xy)^{1/m}$, $q = \ell$ and then substitute it back to (3.2) to get

$$\begin{aligned}\mathcal{H}_0 &= -2\pi^{m/2} x \int_0^\infty (\pi^m xy)^{1/m-1/2} \frac{1}{\sqrt{4\pi^2 m(xy)^{1/m}}} \\ &\quad \times \sum_{j=0}^{\ell} \frac{(\ell+j)!}{j!(\ell-j)!(4\pi m(xy)^{1/m})^j} \\ &\quad \times \left\{ i^{j+q} e(m(xy)^{1/m}) + (-i)^{j+q} e(-m(xy)^{1/m}) \right\} \psi(y) dy.\end{aligned}$$

Denoting

$$a_j^{(0)} = -\frac{1}{\sqrt{m}} \cdot \frac{(\ell+j)!}{j!(\ell-j)!(4\pi m)^j}, \quad j = 0, 1, \dots, \ell,$$

we get

$$\begin{aligned}\mathcal{H}_0 &= x \sum_{j=0}^{\ell} a_j^{(0)} \int_0^{\infty} (xy)^{-(j+\ell)/m} \psi(y) \\ &\quad \times \left\{ i^{j+\ell} e(m(xy)^{1/m}) + (-i)^{j+\ell} e(-m(xy)^{1/m}) \right\} dy.\end{aligned}\tag{3.4}$$

Now we turn to the estimate of \mathcal{H}_1 . We have

$$\begin{aligned}\mathcal{H}_1 &= i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - \ell)}{s\Gamma(-ms + \frac{1}{2})} \\ &\quad \times m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds.\end{aligned}$$

Using the formula $\Gamma(s+1) = s\Gamma(s)$ repeatedly we obtain

$$\begin{aligned}\frac{\Gamma(ms - \ell)}{s} &= m \left\{ \sum_{b=0}^{r-1} \frac{(\ell+b)!}{\ell!} (-1)^b \Gamma(ms - (\ell+b+1)) \right\} \\ &\quad + (-1)^r (\ell+1) \cdots (\ell+r) \frac{\Gamma(ms - (\ell+r))}{s}.\end{aligned}$$

This gives us

$$\begin{aligned}\mathcal{H}_1 &= i\pi^{-m/2-1} \left(\sum_{b=0}^{r-1} \frac{(\ell+b)!}{\ell!} (-1)^b m \mathcal{I}_{b+1} \right) \\ &\quad + i(-1)^r \pi^{-m/2-1} (\ell+1) \cdots (\ell+r) \mathcal{I}_r^*,\end{aligned}\tag{3.5}$$

where for $d = 1, \dots, r$

$$\mathcal{I}_d = \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell+d))}{\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds,\tag{3.6}$$

and

$$\mathcal{I}_r^* = \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell+r))}{s\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds.\tag{3.7}$$

Moving the integral contour in (3.6) to $\operatorname{Re}(s) = -\infty$ and picking up residues $(-1)^n/(mn!)$ at $s = (-n + \ell + d)/m$ for $n = 0, 1, 2, \dots$, we get

$$\begin{aligned}m\mathcal{I}_d &= 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - (\ell+d) + \frac{1}{2})} \\ &\quad \times m^{2(n-\ell-d)+m/2} u^{2(n-\ell-d)/m+1} \tilde{\psi}\left(\frac{2(n-\ell-d)}{m} + 1\right) \\ &= 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - (\ell+d) + \frac{1}{2})} \\ &\quad \times m^{2(n-\ell-d)+m/2} u^{2(n-\ell-d)/m+1} \int_0^{\infty} y^{2(n-\ell-d)/m} \psi(y) dy.\end{aligned}$$

Collecting factors and using (3.1), we have

$$\begin{aligned}
m\mathcal{I}_d &= 2\pi ium^{m/2+1/2-(l+d)} \int_0^\infty (uy)^{1/(2m)-(\ell+d)/m} \\
&\times \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n-(\ell+d)+\frac{1}{2})} (m(uy)^{1/m})^{2n-\ell-d-1/2} \psi(y) dy \\
&= 2\pi ium^{m/2+1/2-(l+d)} \\
&\times \int_0^\infty (uy)^{1/(2m)-(\ell+d)/m} J_{-\ell-d-1/2}(2m(uy)^{1/m}) \psi(y) dy.
\end{aligned}$$

Using (3.3) with $z = 2m(uy)^{1/m} = 2\pi m(xy)^{1/m}$ and $q = \ell + d$ and then following the computation leading to (3.4), we get

$$\begin{aligned}
m\mathcal{I}_d &= ix\pi^{m/2+1-d} m^{1/2-d} \sum_{j=0}^{\ell+d} \frac{(\ell+d+j)!}{j!(\ell+d-j)!(4\pi m)^j} \int_0^\infty (xy)^{-(\ell+d+j)/m} \\
&\times \left\{ i^{\ell+d+j} e(m(xy)^{1/m}) + (-i)^{\ell+d+j} e(-m(xy)^{1/m}) \right\} \psi(y) dy. \tag{3.8}
\end{aligned}$$

To estimate \mathcal{I}_r^* , we move the integral contour in (3.7) from $\text{Re}(s) = \sigma_1(r)$ back to $\text{Re}(s) = \sigma(r) = 1/4 + r/(2m) - \varepsilon$. Then the real part of $ms - (\ell + r)$ moves from ε to $1/4 - (\ell + r)/2 - m\varepsilon$. We pick up residues at $s = (-n + \ell + r)/m$ for $n = 0, \dots, [(\ell + r)/2 - 1/4]$ to get

$$\begin{aligned}
\mathcal{I}_r^* &= 2\pi ium^{m/2} \sum_{n=0}^{[(\ell+r)/2-1/4]} \frac{(-1)^n}{(\ell+r-n)n!\Gamma(n-\ell-r+\frac{1}{2})} \\
&\times \int_0^\infty (m(uy)^{1/m})^{2n-2(\ell+r)} \psi(y) dy \\
&+ \int_{\text{Re } s = \sigma(r)} \frac{\Gamma(ms - (\ell + r))}{s\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds. \tag{3.9}
\end{aligned}$$

Since $\psi(y) (= \phi(y/X))$ is supported on $[aX, bX]$, the first term on the right side above is

$$\ll (uX)^{1+\frac{2[(\ell+r)/2-1/4]-2(\ell+r)}{m}} \ll (xX)^{1-(\ell+r)/m-1/(2m)} = (xX)^{-r/m+1/2},$$

where the implied constant depends on ϕ , a and b . By Stirling's formula, for $\text{Re}(s) = \sigma(r)$ one has

$$\frac{\Gamma(ms - (\ell + r))}{s\Gamma(-ms + \frac{1}{2})} \ll |s|^{2m\sigma(r)-\ell-r-\frac{3}{2}} \ll |s|^{-1-\varepsilon}.$$

By (2.18) the last integral in (3.9) is

$$\ll (uX)^{-2\sigma(r)+1} \ll (xX)^{-r/m+1/2+\varepsilon}.$$

Therefore

$$\mathcal{I}_r^* \ll (uX)^{-2\sigma(r)+1} \ll (xX)^{-r/m+1/2+\varepsilon}. \tag{3.10}$$

This together with (3.8) and (3.5) show that

$$\begin{aligned}
\mathcal{H}_1 &= -xm^{1/2} \sum_{b=0}^{r-1} (\pi m)^{-b-1} \frac{(\ell+b)!}{\ell!} (-1)^b \\
&\times \sum_{j=0}^{\ell+b+1} \frac{(\ell+b+1+j)!}{j!(\ell+b+1-j)!(4\pi m)^j} \int_0^\infty (xy)^{-(\ell+b+1+j)/m} \\
&\times \left\{ i^{\ell+b+1+j} e(m(xy)^{1/m}) + (-i)^{\ell+b+1+j} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\
&+ O((xX)^{-r/m+1/2+\varepsilon}).
\end{aligned}$$

Collecting like terms, we get

$$\begin{aligned}
\mathcal{H}_1 &= x \sum_{t=1}^{\ell+2r} a_t^{(1)} \int_0^\infty (xy)^{-(\ell+t)/m} \\
&\times \left\{ i^{\ell+t} e(m(xy)^{1/m}) + (-i)^{\ell+t} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\
&+ O((xX)^{-r/m+1/2+\varepsilon}),
\end{aligned}$$

where for $t = 1, 2, \dots, \ell + 2r$,

$$a_t^{(1)} = -\frac{4\sqrt{m}}{(4\pi m)^t} \cdot \frac{(\ell+t)!}{\ell!} \sum_{\max\{0, \frac{t-\ell}{2}-1\} \leq b \leq \min(r-1, t-1)} \frac{(-4)^b (\ell+b)!}{(t-b-1)!(\ell+2b-t+2)!}.$$

Note that when $j \geq r+1$,

$$\begin{aligned}
&x \int_0^\infty (xy)^{-(\ell+j)/m} \left\{ i^{\ell+j} e(m(xy)^{1/m}) + (-i)^{\ell+j} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\
&\ll (xX)^{1-(\ell+r+1)/m} = (xX)^{-r/m+1/2-1/(2m)}.
\end{aligned}$$

Thus we finally obtain

$$\begin{aligned}
\mathcal{H}_1 &= x \sum_{t=1}^r a_t^{(1)} \int_0^\infty (xy)^{-(\ell+t)/m} \\
&\times \left\{ i^{\ell+t} e(m(xy)^{1/m}) + (-i)^{\ell+t} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\
&+ O((xX)^{-r/m+1/2+\varepsilon}). \tag{3.11}
\end{aligned}$$

To finish the proof of Theorem 1.1, we need to estimate \mathcal{H}_j for $2 \leq j \leq r$. This is similar to the case when $j = 1$ and so we will briefly describe the idea. By (2.15) we have

$$\mathcal{H}_j = i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - \ell)}{s^j \Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds.$$

By repeated use of the formula $\Gamma(s+1) = s\Gamma(s)$, we have

$$\begin{aligned} \frac{\Gamma(ms - \ell)}{s^j} &= m \left\{ \frac{\Gamma(ms - (\ell + 1))}{s^{j-1}} \right. \\ &\quad + \sum_{k=1}^{h-1} (\ell + 1) \cdots (\ell + k) (-1)^k \frac{\Gamma(ms - (\ell + k + 1))}{s^{j-1}} \Big\} \\ &\quad + (-1)^h (\ell + 1) \cdots (\ell + h) \frac{\Gamma(ms - (\ell + h))}{s^j}, \end{aligned}$$

where $h = r - j + 1$. Applying this process repeatedly, we can finally decompose $\Gamma(ms - \ell)/(s^j \Gamma(-ms + 1/2))$ into finite sums of

$$\lambda \frac{\Gamma(ms - (\ell + p))}{\Gamma(-ms + \frac{1}{2})}, \quad 2 \leq p \leq r,$$

and

$$\mu \frac{\Gamma(ms - (\ell + p))}{s^q \Gamma(-ms + \frac{1}{2})}, \quad p + q = r + 1, \quad 2 \leq p \leq r, \quad 1 \leq q \leq r$$

with some constants $\lambda = \lambda_p, \mu = \mu_{p,q}$. So it remains to estimate integrals of the form

$$\mathcal{I}_p = \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + p))}{\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds, \quad 2 \leq p \leq r,$$

and

$$\mathcal{I}^* = \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + p))}{s^q \Gamma(-ms + \frac{1}{2})} m^{-2ms} u^{-2s} \tilde{\psi}(-2s+1) ds$$

with $p + q = r + 1, \quad 2 \leq p \leq r, \quad 1 \leq q \leq r$.

The integral \mathcal{I}_p has been treated in (3.8). The integral \mathcal{I}^* can be estimated in a similar way as \mathcal{I}_r^* , and it satisfies $\mathcal{I}^* \ll (uX)^{-r/m+1/2+\varepsilon}$. So one finally obtains for $j \geq 2$,

$$\begin{aligned} \mathcal{H}_j &= x \sum_{t=j}^r a_t^{(j)} \int_0^\infty (xy)^{-(\ell+t)/m} \\ &\quad \times \left\{ i^{\ell+t} e(m(xy)^{1/m}) + (-i)^{\ell+t} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\ &\quad + O((xX)^{-r/m+1/2+\varepsilon}). \end{aligned} \tag{3.12}$$

Collecting estimates in (3.4), (3.11) and (3.12), and substituting them into (2.19), we finishes the proof of Theorem 1.1 when m is odd by writing $c_k = \sum_{j=0}^k a_k^{(j)}$. In particular we have $c_0 = a_0^{(0)} = -1/\sqrt{m}$.

4. PROOF OF THEOREM 1.1 WHEN m IS EVEN

By (3.2), when $m = 2l$ one has

$$\mathcal{H}_0 = -2\pi x \int_0^\infty (xy)^{1/m-1/2} \psi(y) J_{-\ell}(2m\pi(xy)^{1/m}) dy. \quad (4.1)$$

Note that for positive integer k , $J_{-k}(z) = (-1)^k J_k(z)$. By (4.8) in [9],

$$\begin{aligned} J_k(z) &= \frac{1}{\sqrt{2\pi z}} e^{i(z-(2k+1)\pi/4)} \sum_{0 \leq j < 2L} \frac{i^j \Gamma(k+j+\frac{1}{2})}{j! \Gamma(k-j+\frac{1}{2}) (2z)^j} (2z)^{-j} \\ &+ \frac{1}{\sqrt{2\pi z}} e^{-i(z-(2k+1)\pi/4)} \sum_{0 \leq j < 2L} \frac{i^j \Gamma(k+j+\frac{1}{2})}{j! \Gamma(k-j+\frac{1}{2}) (2z)^j} (-2z)^{-j} \\ &+ O(|z|^{-2L-1/2}). \end{aligned}$$

Since $e^{-i(2k+1)\pi/4} = (-i)^k i^{-1/2}$ and $e^{i(2k+1)\pi/4} = i^k (-i)^{-1/2}$ the above formula can be rewritten as

$$\begin{aligned} J_{-k}(z) &= \frac{1}{\sqrt{2\pi z}} \sum_{0 \leq j < 2L} \frac{\Gamma(k+j+\frac{1}{2})}{j! \Gamma(k-j+\frac{1}{2}) (2z)^j} \\ &\times \left\{ i^{j+k-1/2} e\left(\frac{z}{2\pi}\right) + (-i)^{j+k-1/2} e\left(-\frac{z}{2\pi}\right) \right\} \\ &+ O(|z|^{-2L-1/2}). \end{aligned} \quad (4.2)$$

Let $z = 2m\pi(xy)^{1/m}$ and $k = \ell$ in (4.2), we get

$$\begin{aligned} \mathcal{H}_0 &= -\frac{x}{\sqrt{m}} \sum_{0 \leq j < 2L} \frac{\Gamma(\ell+j+\frac{1}{2})}{j! \Gamma(\ell-j+\frac{1}{2}) (4\pi m)^j} \int_0^\infty (xy)^{1/(2m)-(j+\ell)/m} \psi(y) \\ &\times \left\{ i^{j+\ell-1/2} e(m(xy)^{1/m}) + (-i)^{j+\ell-1/2} e(-m(xy)^{1/m}) \right\} dy \\ &+ O((xX)^{1/2-(2L)/m+1/(2m)}). \end{aligned}$$

For any $r \geq 2$ and $L = [r/2] + 1$, the error term above is $O((xX)^{1/2-r/m})$, and this gives

$$\begin{aligned} \mathcal{H}_0 &= x \sum_{j=0}^{2[r/2]+1} b_j^{(0)} \int_0^\infty (xy)^{1/(2m)-(j+\ell)/m} \psi(y) \\ &\times \left\{ i^{j+\ell-1/2} e(m(xy)^{1/m}) + (-i)^{j+\ell-1/2} e(-m(xy)^{1/m}) \right\} dy \\ &+ O((xX)^{1/2-r/m}), \end{aligned}$$

where

$$b_j^{(0)} = -\frac{1}{\sqrt{m}} \cdot \frac{\Gamma(\ell+j+\frac{1}{2})}{j! \Gamma(\ell-j+\frac{1}{2}) (4\pi m)^j}.$$

Now we turn to the estimate of \mathcal{H}_j for $j \geq 1$. By (2.15) we have

$$\mathcal{H}_j = i\pi^{-\ell-1} \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell - \frac{1}{2}))}{s^j \Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds.$$

We first consider \mathcal{H}_1 . Using the formula $\Gamma(s+1) = s\Gamma(s)$, one obtains

$$\frac{\Gamma(ms - (\ell - \frac{1}{2}))}{s} = m\Gamma\left(ms - \left(\ell + \frac{1}{2}\right)\right) - \left(\ell + \frac{1}{2}\right) \frac{\Gamma(ms - (\ell + \frac{1}{2}))}{s}.$$

Applying this formula to the quotient on its right side repeatedly, we get

$$\begin{aligned} & \frac{\Gamma(ms - (\ell - \frac{1}{2}))}{s} \\ &= m\left\{\Gamma\left(ms - \left(\ell + \frac{1}{2}\right)\right) \right. \\ &+ \sum_{b=1}^{r-1} (-1)^b \left(\ell + \frac{1}{2}\right) \cdots \left(\ell + \frac{2b-1}{2}\right) \Gamma\left(ms - \left(\ell + \frac{2b+1}{2}\right)\right) \Big\} \\ &+ (-1)^r \left(\ell + \frac{1}{2}\right) \cdots \left(\ell + \frac{2r-1}{2}\right) \frac{\Gamma(ms - (\ell + \frac{2r-1}{2}))}{s}. \end{aligned}$$

Substituting this into the integral defining \mathcal{H}_1 , we have

$$\begin{aligned} \mathcal{H}_1 &= \frac{i\pi^{-\ell-1}}{\Gamma(\ell + \frac{1}{2})} \sum_{b=0}^{r-1} (-1)^b \Gamma\left(\ell + b + \frac{1}{2}\right) m \mathfrak{J}_{b+1} \\ &+ i\pi^{-\ell-1} (-1)^r \left(\ell + \frac{1}{2}\right) \cdots \left(\ell + r - \frac{1}{2}\right) \mathfrak{J}_r^*, \end{aligned} \tag{4.3}$$

where for $d = 1, \dots, r$

$$\mathfrak{J}_d = \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + d - \frac{1}{2}))}{\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds,$$

and

$$\mathfrak{J}_r^* = \int_{\operatorname{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + r - \frac{1}{2}))}{s \Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds.$$

For \mathfrak{J}_d we move the integral line to $\Re s = -\infty$ and pick up residues $(-1)^n/(n!m)$ at $s = (-n + \ell + d - 1/2)/m$ with $n = 0, 1, 2, \dots$. Consequently,

$$\begin{aligned}
m\mathfrak{J}_d &= 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^n m^{2(n-\ell-d)+m/2+1}}{n! \Gamma(n+1-\ell-d)} \\
&\times u^{2(n-\ell-d)/m+1/m+1} \tilde{\psi}\left(\frac{2(n-\ell-d)}{m} + \frac{1}{m} + 1\right) \\
&= 2\pi i u m^{1-d} \int_0^{\infty} (uy)^{1/m-(\ell+d)/m} \\
&\times \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+1-(\ell+d))} (m(uy)^{1/m})^{2n-(\ell+d)} \psi(y) dy.
\end{aligned}$$

By the series definition of the Bessel-function in (3.1), we have

$$m\mathfrak{J}_d = 2\pi i u m^{1-d} \int_0^{\infty} (uy)^{1/m-(\ell+d)/m} J_{-(\ell+d)}(2m(uy)^{1/m}) \psi(y) dy.$$

Applying the asymptotic expansion in (4.2) with $z = 2m(uy)^{1/m} = 2\pi m(xy)^{1/m}$, $k = \ell + d$, we get

$$\begin{aligned}
m\mathfrak{J}_d &= i\pi^{1+\ell-d} m^{1/2-d} x \sum_{0 \leq j < 2L} \frac{\Gamma(\ell + d + j + \frac{1}{2})}{j! \Gamma(\ell + d - j + \frac{1}{2}) (4\pi m)^j} \\
&\times \int_0^{\infty} (xy)^{1/(2m)-(\ell+d+j)/m} \\
&\times \left\{ i^{j+\ell+d-1/2} e(m(xy)^{1/m}) + (-i)^{\ell+d+j-1/2} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\
&+ O((xX)^{-2L/m+1/(2m)-d/m+1/2}).
\end{aligned}$$

Let $L = [(r-d)/2] + 1$. Since $L \geq (r-d)/2 + 1/2$, the error term above is $O((xX)^{-r/m+1/2})$, and this finishes the estimate for $m\mathfrak{J}_d$.

To estimate \mathfrak{J}_r^* , we move the integral contour from $\Re(s) = \sigma_1(r)$ in (2.17) to $\Re(s) = \sigma(r) = 1/4 + r/(2m) - \varepsilon$ in (2.5). Then $\Re(ms - (\ell + r - 1/2))$ moves from $m\varepsilon$ to $-(\ell + r - 1)/2 - m\varepsilon$. Picking up residues in this region at $ms - (\ell + d - 1/2) = -n$ for $0 \leq n \leq (\ell + r - 1)/2$, we get

$$\begin{aligned}
\mathfrak{J}_r^* &= 2\pi i u \sum_{n=0}^{[(\ell+r-1)/2]} \frac{(-1)^n m^{2(n-r)-\ell+1}}{(\ell + r - n - \frac{1}{2}) n! \Gamma(n - \ell - r + 1)} \\
&\times \int_0^{\infty} (uy)^{1/m+2(n-\ell-r)/m} \psi(y) dy \\
&+ \int_{\Re s = \sigma(r)} \frac{\Gamma(ms - (\ell + r - \frac{1}{2}))}{s \Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s+1) ds. \quad (4.4)
\end{aligned}$$

By Stirling's formula, for $\operatorname{Re} s = \sigma(r)$ one has

$$\frac{\Gamma(ms - (\ell + r - \frac{1}{2}))}{s\Gamma(-ms + \frac{1}{2})} \ll |s|^{2m\sigma(r) - \ell - r - 1} \ll |s|^{-1-\varepsilon}.$$

By (2.18) the last integral in (4.4) is $O((xX)^{-r/m+1/2+2\varepsilon})$. The first quantity on the right of (4.4) is

$$\ll (uX)^{1+\frac{1}{m}+\frac{2[(\ell+r-1)/2]-2\ell-2r}{m}} \ll (xX)^{-r/m+1/2}.$$

Back to (4.3) and collecting our results on $m\mathfrak{I}_d$, \mathfrak{I}_r^* and their error terms, we get

$$\begin{aligned} \mathcal{H}_1 &= -\frac{\sqrt{m}x}{\Gamma(\ell + \frac{1}{2})} \sum_{b=0}^{r-1} (-1)^b \Gamma\left(\ell + b + \frac{1}{2}\right) (\pi m)^{-b-1} \\ &\times \sum_{j=0}^{2[(r-b-1)/2]+1} \frac{\Gamma(\ell + b + 1 + j + \frac{1}{2})}{j! \Gamma(\ell + b + 1 - j + \frac{1}{2}) (4\pi m)^j} \int_0^\infty (xy)^{1/(2m) - (\ell+b+1+j)/m} \\ &\times \left\{ i^{\ell+b+j+1/2} e(m(xy)^{1/m}) + (-i)^{\ell+b+j+1/2} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\ &+ O((xX)^{-r/m+1/2+\varepsilon}). \end{aligned}$$

We write $t = b + j + 1$ and collect like terms with coefficients

$$\begin{aligned} b_t^{(1)} &= -\frac{4\sqrt{m}}{(4\pi m)^t} \cdot \frac{\Gamma(\ell + t + \frac{1}{2})}{\Gamma(\ell + \frac{1}{2})} \sum_{0 \leq b < \min(r,t)} \frac{(-4)^b \Gamma(\ell + b + \frac{1}{2})}{\Gamma(t-b) \Gamma(\ell + 2b - t + \frac{5}{2})}, \\ t &= 1, 2, \dots, r+1. \end{aligned}$$

Then we obtain

$$\begin{aligned} \mathcal{H}_1 &= x \sum_{t=1}^{r+1} b_t^{(1)} \int_0^\infty (xy)^{1/(2m) - (t+\ell)/m} \\ &\times \left\{ i^{\ell+t-1/2} e(m(xy)^{1/m}) + (-i)^{\ell+t-1/2} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\ &+ O((xX)^{-r/m+1/2+\varepsilon}). \end{aligned}$$

One can estimates \mathcal{H}_j ($j \geq 2$) in a similar way, and finally obtain

$$\begin{aligned} \mathcal{H}_j &= x \sum_{t=j}^{r+1} b_t^{(j)} \int_0^\infty (xy)^{1/(2m) - (t+\ell)/m} \\ &\times \left\{ i^{\ell+t-1/2} e(m(xy)^{1/m}) + (-i)^{\ell+t-1/2} e(-m(xy)^{1/m}) \right\} \psi(y) dy \\ &+ O((xX)^{-r/m+1/2+\varepsilon}). \end{aligned}$$

Substituting these estimates into (2.19) and putting the term with $t = r + 1$ into the error term we finish the proof of Theorem 1.1 for even m by letting $c_k = \sum_{j=0}^k b_k^{(j)}$ where $c_0 = b_0^{(0)} = -1/\sqrt{m}$.

5. PROOF OF THEOREM 1.2

By (1.9) with $\psi(x) = \phi(x/X)e(\alpha x^\beta)$, we get

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \phi\left(\frac{|n|}{X}\right) = \sum_{n \neq 0} \frac{A_f(1, \dots, 1, n)}{|n|} \Phi(|n|), \quad (5.1)$$

where by Theorem 1.1, for $r > m/2$ one has

$$\begin{aligned} \Phi(x) &= x \sum_{k=0}^r \tilde{c}_k \sum_{\pm} (\pm i)^{k+(m-1)/2} \int_0^\infty (xy)^{1/(2m)-1/2-k/m} \phi\left(\frac{y}{X}\right) e(\alpha y^\beta \pm m(xy)^{1/m}) dy \\ &+ O((xX)^{-r/m+1/2+\varepsilon}). \end{aligned}$$

Making change of variable $y = t^m X$ and putting in (5.1), we get

$$\begin{aligned} &\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \phi\left(\frac{|n|}{X}\right) \\ &= m \sum_{k=0}^r \tilde{c}_k X^{1/(2m)+1/2-k/m} \sum_{n=1}^\infty \frac{A_f(1, \dots, 1, n) + A_f(1, \dots, 1, -n)}{n^{1/2+k/m-1/(2m)}} \\ &\quad \times \sum_{\pm} (\pm i)^{k+(m-1)/2} I_k(n; \pm) \\ &+ O\left(X^{-r/m+1/2+\varepsilon} \sum_{n=1}^\infty \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{r/m+1/2-\varepsilon}}\right), \end{aligned} \quad (5.2)$$

where

$$I_k(n; \pm) = \int_0^\infty t^{m/2-k-1/2} \phi(t^m) e(\alpha X^\beta t^{m\beta} \pm (nX)^{1/m} mt) dt. \quad (5.3)$$

By Rankin-Selberg method for $GL(n) \times GL(n)$ convolution [26] (Remark 12.1.8), one has

$$\sum_{1 \leq |n| \leq X} |A_f(1, \dots, 1, n)|^2 \ll X. \quad (5.4)$$

Therefore the O -term in (5.2) is $O(X^{-r/m+1/2+\varepsilon})$ for $r > m/2$, where the implied constant depends on r, m and ε . To estimate the integral in (5.3) we consider integral of the form

$$\int_0^\infty h(t) e(f(t)) dt,$$

where $h, f \in C^\infty(\mathbb{R})$ and h is supported on $[a, b] \subset (0, \infty)$. Suppose $f'(x) \neq 0$ for $x \in [a, b]$. By repeated partial integrating by parts, one obtains for $j \geq 0$ that

$$\int_0^\infty h(t)e(f(t))dt = \left(\frac{-1}{2\pi i}\right)^j \int_0^\infty h_j(t)e(f(t))dt, \quad (5.5)$$

where $h_0(t) = h(t)$ and

$$h_j(t) = \left(\frac{h_{j-1}(t)}{f'(t)}\right)' := \frac{g_j(t)}{(f'(t))^{2j}}, \quad j \geq 1.$$

Let $h(t) = t^{m/2-k-1/2}\phi(t^m)$ and $f(t) = \alpha X^\beta t^{m\beta} + m(nX)^{1/m}t$ in (5.5) one easily obtains

$$I_k(n; +) \ll_{m,j} (nX)^{-j/m}, \quad \text{for } n \geq 1.$$

Set $j = r + 1$. Then the contribution of the terms in \sum_+ to (5.2) is

$$\ll X^{-1/(2m)+1/2-r/m} \sum_{n=1}^\infty \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{r/m+1/2+1/(2m)}} \ll_{r,m} X^{-1/(2m)+1/2-r/m}.$$

Here we have used (5.4). To estimate the contribution of the terms concerning \sum_- , we write

$$n_0 = \frac{1}{2} \min\{1, 2^{\beta-1/m}\}(\alpha\beta X^\beta)^m X^{-1}, \quad (5.6)$$

$$n_1 = 2 \max\{1, 2^{\beta-1/m}\}(\alpha\beta X^\beta)^m X^{-1}. \quad (5.7)$$

Then for $n \notin (n_0, n_1)$ one has $I_k(n; -) \ll (nX)^{-j/m}$. Therefore the contribution of the terms with $n \notin (n_0, n_1)$ in \sum_- is $O(X^{-1/(2m)+1/2-r/m})$. This shows that

$$\begin{aligned} & \sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha|n|^\beta) \phi\left(\frac{|n|}{X}\right) \\ &= m \sum_{k=0}^r \tilde{c}_k(-i)^{k+(m-1)/2} X^{1/(2m)+1/2-k/m} \\ & \quad \times \sum_{n_0 < n < n_1} \frac{A_f(1, \dots, 1, n) + A_f(1, \dots, 1, -n)}{n^{1/2+k/m-1/(2m)}} I_k(n; -) \\ & \quad + O_{r,m,\varepsilon}\left(X^{-r/m+1/2+\varepsilon}\right). \end{aligned} \quad (5.8)$$

If $2 \max\{1, 2^{\beta-1/m}\}(\alpha\beta)^m \leq X^{1-\beta m}$, then $n_1 \leq 1$. Hence the main term in (5.8) disappears and the estimate

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha|n|^\beta) \phi\left(\frac{|n|}{X}\right) \ll_{m,\beta,r} X^{-r/m+1/2+\varepsilon} \ll_{m,\beta,M} X^{-M}$$

holds for any $M > 0$ by taking r sufficiently large in terms of M . This proves Theorem 1.2 (i).

If $2 \max\{1, 2^{\beta-1/m}\}(\alpha\beta)^m > X^{1-\beta m}$, then $n_1 > 1$. We distinguish two cases according to $\beta \neq 1/m$ or not. For $\beta \neq 1/m$ we have

$$(\alpha X^\beta t^{m\beta} - (nX)^{1/m} mt)'' = \alpha(m\beta)(m\beta - 1)X^\beta t^{m\beta-2} \gg_{m,\beta} \alpha X^\beta.$$

By the second derivative test one has $I_k(n; -) \ll_{\beta,m} (\alpha X^\beta)^{-1/2}$. Thus the main term in (5.8) is

$$\begin{aligned} & \ll_{m,\beta} X^{1/(2m)+1/2} (\alpha X^\beta)^{-1/2} \sum_{n_0 < n < n_1} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2-1/(2m)}} \\ & \ll_{m,\beta} (n_1 X)^{1/(2m)+1/2} (\alpha X^\beta)^{-1/2} \ll_{m,\beta} (\alpha X^\beta)^{m/2}. \end{aligned}$$

Choosing $r = [(m+1)/2]$ we get

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha n^\beta) \phi\left(\frac{|n|}{X}\right) \ll_{m,\beta} (\alpha X^\beta)^{m/2}.$$

For $\beta = 1/m$, we use the obvious estimate $I_k(n; -) \ll 1$ in (5.8) to get

$$\sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \phi\left(\frac{|n|}{X}\right) \ll (n_1 X)^{1/2+1/(2m)} \ll_m (\alpha X^\beta)^{(m+1)/2}.$$

This proves (1.14).

Moreover, when $\beta = 1/m$, one has $I = (n_0, n_1) = ((\alpha/m)^m/2, 2(\alpha/m)^m)$ with $2(\alpha/m)^m \geq 1$, and

$$I_k(n; -) = \int_0^\infty t^{m/2-k-1/2} \phi(t^m) e((\alpha - mn^{1/m})X^{1/m}t) dt.$$

Since $(\alpha/m)^m > 1/2$, there is a unique integer $n_\alpha \geq 1$ such that

$$(\alpha/m)^m = n_\alpha + \lambda, \quad -1/2 < \lambda \leq 1/2.$$

For $n \in I$, $n \neq n_\alpha$, one has $|n^{1/m} - \alpha/m| \gg_m |n - n_\alpha| \alpha^{1-m}$. By repeated partial integrating by parts we get

$$I_k(n; -) \ll_{m,j} \frac{1}{(|n - n_\alpha| \alpha^{1-m} X^{1/m})^j}, \quad j \geq 0.$$

Putting in (5.8) and applying (5.4), the main terms except the term with $n = n_\alpha$ produce the contribution which is, for $j \geq 1$,

$$\begin{aligned} & \ll_m X^{1/(2m)+1/2} (\alpha^{m-1} X^{-1/m})^j \sum_{\substack{n_0 < n < n_1 \\ n \neq n_\alpha}} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2-1/(2m)}} \cdot \frac{1}{|n - n_\alpha|^j} \\ & \ll_m X^{1/(2m)+1/2} (\alpha^{m-1} X^{-1/m})^j n_1^{1/(2m)} \ll_m X^{1/(2m)+1/2} (\alpha^{m-1} X^{-1/m})^j \alpha^{1/2}. \end{aligned} \quad (5.9)$$

Here we have used (5.4). Let $0 < \varepsilon < 1/m$. If $X > \alpha^{m(m-1)/(1-m\varepsilon)}$, one has $\alpha^{m-1} X^{-1/m} < X^{-\varepsilon}$. Thus the last expression in (5.9) is $\ll X^{-r/m+1/2+\varepsilon}$ by taking j sufficiently large in

terms of r . This proves

$$\begin{aligned}
& \sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \phi\left(\frac{|n|}{X}\right) \\
&= m \{A_f(1, \dots, 1, n_\alpha) + A_f(1, \dots, 1, -n_\alpha)\} \sum_{k=0}^r \rho_+(k, m, \alpha, X) X^{1/(2m)+1/2-k/m} \\
&+ O_{r,m,\varepsilon}(X^{-r/m+1/2+\varepsilon})
\end{aligned} \tag{5.10}$$

with

$$\rho_+(k, m, \alpha, X) = \tilde{c}_k(-i)^{k+(m-1)/2} \frac{I_k(n; -)}{n_\alpha^{1/2+k/m-1/(2m)}}.$$

In particular, suppose $(\alpha/m)^m = q$ is an integer, that is $\alpha = mq^{1/m}$. Then $n_\alpha = q$ and

$$I_k(n; -) = \int_0^\infty t^{m/2-k-1/2} \phi(t^m) = \frac{1}{m} \int_0^\infty x^{1/(2m)-1/2-k/m} \phi(x) dx.$$

This finishes the proof of Theorem 1.2 and Corollary 1.1 with the exponential function being $e(\alpha|n|^\beta)$. Proof for the case of $e(-\alpha|n|^\beta)$ is analogous.

6. PROOF OF THEOREMS 1.3 AND 1.4

Let $\Delta > 1$ and $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ be a C^∞ function supported on $[1 - \Delta^{-1}, 2 + \Delta^{-1}]$ such that $\phi(x) \equiv 1$ for $x \in [1, 2]$ and satisfies

$$\phi^{(j)}(x) \ll \Delta^j, \quad \text{for any integer } j \geq 0. \tag{6.1}$$

Then by (5.4) and Cauchy's inequality we get

$$\begin{aligned}
& \sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \\
&= \sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \phi\left(\frac{|n|}{X}\right) + O(X\Delta^{-1/2}).
\end{aligned} \tag{6.2}$$

By (5.2) and applying (5.4) again we have

$$\begin{aligned}
& \sum_{n \neq 0} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \phi\left(\frac{|n|}{X}\right) \\
&= m \sum_{k=0}^r \tilde{c}_k X^{1/(2m)+1/2-k/m} \sum_{n>0} \frac{A_f(1, \dots, 1, n) + A_f(1, \dots, 1, -n)}{n^{1/2+k/m-1/(2m)}} \\
&\quad \times \sum_{\pm} (\pm i)^{k+(m-1)/2} I_k(n; \pm) + O_m(1),
\end{aligned} \tag{6.3}$$

where $r = [(m+1)/2]$ and $I_k(n; \pm)$ is defined as in (5.3). By (5.5) and (6.1), for $j \geq 1$ we have

$$I_k(n; +) \ll_{m,k,j} (nX)^{-j/m} \Delta^{j-1}, \quad \text{for } n \geq 1.$$

Set $j = r$ for $n > H = \Delta^m X^{-1}$ and $j = 1$ for $n \leq H$. The contribution of $I(n; +)$ to (6.3) is

$$\begin{aligned}
&\ll mX^{1/2-1/(2m)} \sum_{n \leq H} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2+1/(2m)}} \\
&\quad + mX^{1/(2m)+1/2-r/m} \Delta^{r-1} \sum_{n > H} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2+r/m-1/(2m)}} \\
&\ll m(XH)^{1/2-1/(2m)} + m(XH)^{1/(2m)+1/2-r/m} \Delta^{r-1} \\
&\ll m\Delta^{(m-1)/2}.
\end{aligned}$$

Next, let n_1 be given by (5.7). Then for $j \geq 1$,

$$I_k(n; -) \ll_{m,k,j} (nX)^{-j/m} \Delta^{j-1}, \quad \text{for } n \geq n_1. \quad (6.4)$$

Applying (6.4) with $j = 1$ for $n_1 \leq n \leq H = \Delta^m X^{-1}$ and $j = r$ for $n > H$, then the contribution of $I_k(n; -)$ with $n \geq n_1$ to (6.3) is $O(\Delta^{(m-1)/2})$. This together with (6.2) shows that

$$\begin{aligned}
&\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha|n|^\beta) \\
&= m \sum_{k=0}^r \tilde{c}_k X^{1/(2m)+1/2-k/m} \sum_{1 \leq n < n_1} \frac{A_f(1, \dots, 1, n) + A_f(1, \dots, 1, -n)}{n^{1/2+k/m-1/(2m)}} \\
&\quad \times \sum_{\pm} (\pm i)^{k+(m-1)/2} I_k(n; \pm) + O(\Delta^{(m-1)/2}) + O(X\Delta^{-1/2}). \quad (6.5)
\end{aligned}$$

Suppose that the parameters α, β, X satisfy (1.11). Then $n_1 < 1$ and the main term above disappears. Setting $\Delta = X^{2/m}$ we get

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha|n|^\beta) \ll_m \Delta^{(m-1)/2} + \Delta X^{-1/2} \ll_m X^{1-1/m}.$$

Suppose that the parameters α, β, X satisfy (1.13), then $n_1 > 1$. Now we have

$$I_k(n; -) = \int_1^{2^{1/m}} t^{m/2-k-1/2} e(\alpha X^\beta t^{m\beta} - (nX)^{1/m} mt) dt + O(\Delta^{-1}). \quad (6.6)$$

For $\beta \neq 1/m$, the second derivative test shows that

$$I_k(n; -) \ll_{m,\beta,k} (\alpha X^\beta)^{-1/2} + \Delta^{-1}.$$

Thus the main term of (6.5) is

$$\begin{aligned}
&\ll X^{1/2+1/(2m)} \left(\Delta^{-1} + (\alpha X^\beta)^{-1/2} \right) \sum_{1 \leq n \leq n_1} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2-1/(2m)}} \\
&\ll (Xn_1)^{1/(2m)+1/2} \left(\Delta^{-1} + (\alpha X^\beta)^{-1/2} \right) \\
&\ll (\alpha X^\beta)^{(m+1)/2} \Delta^{-1} + (\alpha X^\beta)^{m/2}.
\end{aligned}$$

Setting $\Delta = \max\{\alpha X^\beta, X^{2/m}\}$, we yield

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \ll (\alpha X^\beta)^{m/2} + X^{1-1/m}.$$

For $\beta = 1/m$, we let $n_\alpha \geq 1$ be the integer such that $(\alpha/m)^m - n_\alpha \in (-1/2, 1/2]$. By (6.6) for $n \neq n_\alpha$,

$$I_k(n; -) \ll_m \frac{1}{|n - n_\alpha| \alpha^{1-m} X^{1/m}} + O(\Delta^{-1}).$$

Choosing $\Delta = \max\{\alpha X^\beta, X^{2/m}\}$, the contribution of the terms with $n \neq n_\alpha$ in (6.5) is

$$\begin{aligned} & \ll X^{1/2-1/(2m)} \alpha^{(m-1)} \sum_{\substack{1 \leq n \leq n_1 \\ n \neq n_\alpha}} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2-1/(2m)}} \cdot \frac{1}{|n - n_\alpha|} \\ & + X^{1/(2m)+1/2} \Delta^{-1} \sum_{\substack{1 \leq n \leq n_1 \\ n \neq n_\alpha}} \frac{|A_f(1, \dots, 1, n)| + |A_f(1, \dots, 1, -n)|}{n^{1/2-1/(2m)}} \\ & \ll \alpha^{m-1/2} X^{1/2-1/(2m)} + (n_1 X)^{1/2+1/(2m)} \Delta^{-1} \\ & \ll \alpha^{m-1/2} X^{1/2-1/(2m)}. \end{aligned}$$

Thus we get

$$\begin{aligned} & \sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \\ & = m \tilde{c}_0 X^{1/(2m)+1/2} \frac{A_f(1, \dots, 1, n_\alpha) + A_f(1, \dots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} (-i)^{(m-1)/2} I_0(n_\alpha; -) \\ & + O_m(\alpha^{m-1/2} X^{1/2-1/(2m)}) + O_m(X^{1-1/m}). \end{aligned} \quad (6.7)$$

By (5.4) the above fraction is $O(\alpha^{1/2})$. Therefore the main term in (6.7) is $O(\alpha^{1/2} X^{1/(2m)+1/2})$. This proves (1.18) and hence finishes the proof of Theorem 1.3.

Now we assume the following

$$A_f(1, \dots, 1, n) \ll n^\theta, \quad \text{for some } 0 < \theta < \frac{2}{m-1}.$$

Then the error term $O(X \Delta^{-1/2})$ in (6.2) and (6.5) becomes $O(X^{1+\theta} \Delta^{-1})$. Suppose that the parameters α, β, X satisfy (1.11), then we can take $\Delta = X^{2(1+\theta)/(m+1)}$ and get

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \ll_m X^{(m-1)(1+\theta)/(m+1)}.$$

Suppose the parameters α, β, X satisfy (1.13), then we can set $\Delta = \max\{\alpha X^\beta, X^{2(1+\theta)/(m+1)}\}$ to yield, for $\beta \neq 1/m$,

$$\sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \ll (\alpha X^\beta)^{m/2} + X^{(m-1)(1+\theta)/(m+1)},$$

and for $\beta = 1/m$,

$$\begin{aligned}
& \sum_{X < |n| \leq 2X} A_f(n, 1, \dots, 1) e(\alpha |n|^\beta) \\
&= m\tilde{c}_0 X^{1/(2m)+1/2} \frac{A_f(1, \dots, 1, n_\alpha) + A_f(1, \dots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} (-i)^{(m-1)/2} I_0(n_\alpha; -) \\
&+ O_m(\alpha^{m-1/2} X^{1/2-1/(2m)}) + O_m(X^{(m-1)(1+\theta)/(m+1)}). \tag{6.8}
\end{aligned}$$

Note that $I_0(n_\alpha; -) = I(m, \alpha, X) + O(\Delta^{-1})$, where

$$I(m, \alpha, X) = \int_1^{2^{1/m}} t^{m/2-1/2} e((\alpha - mn_\alpha^{1/m})X^{1/m}t) dt.$$

Thus the main term in (6.8) can be rewritten as

$$m\tilde{c}_0 X^{1/(2m)+1/2} \frac{A_f(1, \dots, 1, n_\alpha) + A_f(1, \dots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} (-i)^{(m-1)/2} I(m, \alpha, X).$$

This finishes the proof of Theorem 1.4.

7. CONCLUSION AND DISCUSSION

Two important features of smoothly weighted sums of Fourier coefficients of a Maass form for $GL_m(\mathbb{Z})$ against $e(\alpha n^\beta)$ have been discovered. They are rapid decay and resonance for various combinations of α and β . These features capture the vibration behavior of the Fourier coefficients of a Maass form. It is interesting to see whether these two features can be used to characterize Fourier coefficients of Maass forms. On the other hand, when β is large, our methods failed to derive a non-trivial bound for the smoothly weighted sum. These are subjects of our subsequent research.

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